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Best sextic approximation of circular arcs with thirteen equioscillations

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Abstract: We approximate a circular arc using a polynomial curve of degree 6. The approximation has least deviation from the x-axis and the error function is of degree 12; the error function equioscillates 13 times rather than the classical 8 times equioscillations that are mathematically guaranteed by the Borel and Chebyshev theorems without a method to find their approximation.

keywords: Bézier curves; sextic approximation; circular arc; equioscillation; CAD.

MSC: 41A10; 41A25; 41A50; 65D10; 65D17.

1 Introduction

The set $C[a,b]$ contains continuous functions on the closed interval $[a,b]$. The uniform (Chebyshev) norm on the linear space $C[a,b]$ is defined by

$$||f||_\infty = \max_{a\le x\le b} |f(x)|, \quad \forall f \in C[a,b].$$

The set $\mathbf{P_n}$ contains all polynomials of degree n. Define the deviation $\triangle(P_n)$ of a polynomial $P_n \in \mathbf{P_n}$ from the function $f \in C[a,b]$ by

$$\triangle(P_n) = ||f - P_n||_\infty.$$

For $n \geq 0$, we define

$$E_n = E_n(f) = \inf\{\triangle(P_n), \forall P_n \in \mathbf{P_n}\}.$$

The set E_n satisfies

$$E_n \geq 0, \quad E_0 \geq E_1 \geq E_2 \geq E_3 \geq \cdots .$$

E_n is closed and, therefore, is compact.

Consider a given function $f \in C[a,b]$, then a polynomial P_n^* is called the polynomial of best uniform approximation to f if $\triangle(P_n^*) = E_n$.

The existence of a polynomial P_n^* in $\mathbf{P_n}$ for which $\triangle(P_n^*) = E_n$ is proved by E. Borel, see [9]. The equioscillation of $n+2$ times is used. A function $E(x)$ is said to equioscillate $n+2$ times on $[a,b]$ if there are $n+2$ points, $a \leq x_1 < x_2 < \cdots < x_{n+1} < x_{n+2} \leq b$ with

$$|E(x_i)| = \max_{a \leq x \leq b} |E(x)|, \ 1 \leq i \leq n+2$$

and $E(x_{i+1})$ has opposite sign of $E(x_i)$, $1 \leq i \leq n+1$. The uniqueness of this polynomial P_n^* and the characterization are proved by Chebyshev by showing the existence of $n+2$ points (Chebyshev alternant, $x_1 < x_2 < \cdots < x_{n+2}$) which alternately satisfy

$$P_n^*(x_i) - f(x_i) = \mp E_n, \ i = 1, 2, \ldots, n+2.$$

Although, theoretically, there is a unique solution; but, for computational purposes, we can only find polynomials of degrees 0 and 1. There is no method to find polynomials of best approximation of degrees $n \geq 2$. The improvement in this field is very slow and it is a challenging issue to tackle this problem; "As a matter of fact, the latter problem involves such formidable difficulties that a general solution has not been found to this day [9]".

The set $\tilde{\mathbf{P}}_n$ contains all monic polynomials of degree n on $I = [0,1]$. It is well-known that the shifted monic Chebyshev polynomial $\tilde{T}_n(2t-1)$ has the smallest maximum value on I, i.e.

$$||\tilde{T}_n||_\infty \leq ||\tilde{P}_n||_\infty, \quad \forall \tilde{P}_n \in \tilde{\mathbf{P}}_n,$$

where equality holds only if $\tilde{P}_n = \tilde{T}_n$. The Chebyshev polynomial $\tilde{T}_n(2t-1)$ equioscillates $n+2$ times in $[0,1]$.

In this paper, the circlular arc is approximated. We find the polynomial of degree six of best uniform approximation that equioscillates 13 times rather than 8 times. This is a substantial improvement because finding the polynomial of best approximation of degree six that equioscillates 8 times is not possible so far.

Parametric curves are flexibil in representing and building curves. In our case, they offer additional degrees of freedom that are used to better represent the original curve. This property is used in [10, 11] to improve the approximation order by polynomials of degree n from $n+1$ to $2n$.

This paper is organized as follows. Preliminaries are given in section 2. The Bézier points and curve are given in section 3. The sextic Bézier curve of best uniform approximation is presented and proved in section 4, and the properties are presented in section 5.

2 Preliminaries

A circular arc $c : t \mapsto (\cos(t), \sin(t))\,,\ -\theta \le t \le \theta$, given in Fig. 1, needs to be approximated by a polynomial curve with best uniform approximation. We use the geometric symmetries of the circle to choose in a proper way the Bézier points. The required sextic Bézier curve has to intersect the circular arc 12 times. The associated error function has to equioscillate 13 times.

The circle c is approximated in this paper using a sextic parametrically defined polynomial curve $p : t \mapsto (x(t), y(t))\,,\ 0 \le t \le 1$, where $x(t), y(t)$ are polynomials of degree 6, that approximates c with "minimum" error. Many researchers have tackled this issue using different degrees, norms, and methods, see [1, 2, 3, 4, 6, 8, 13, 12]. The results of our method in this paper are optimal and can not be improved.

The proper distance function to measure the error between p and c is the Euclidean error function:

$$E(t) := \sqrt{x^2(t) + y^2(t)} - 1. \tag{1}$$

$E(t)$ will be replaced by the following error function

$$e(t) := x^2(t) + y^2(t) - 1. \tag{2}$$

Since $e(t) = 0$ is the implicit equation of the unit circle; this implies that the $e(t)$ error function is a suitable measure to test if $x(t)$ and $y(t)$ satisfy this equation and

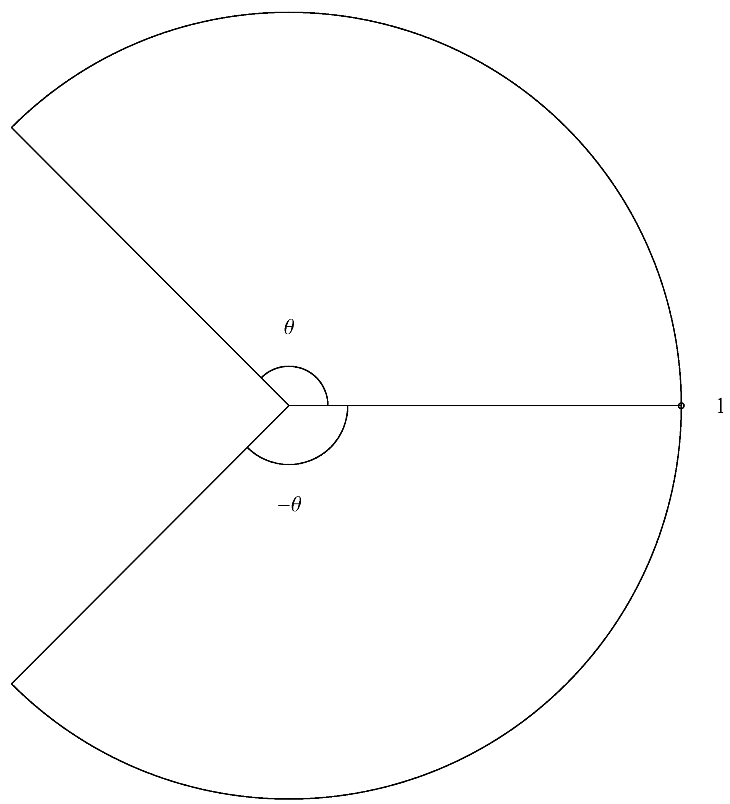

Figure 1: A circular arc

to measure the error. Both error functions share the same roots and critical points, see Propositions I and II.

The approximation problem: Find a polynomial curve $p : t \mapsto (x(t), y(t))$, $0 \leq t \leq 1$, where $x(t), y(t)$ are of degree 6, that approximates c and satisfies the following requirements:

1. p minimizes $\max_{t\in[0,1]} \ |e(t)|$,
2. $e(t)$ equioscillates 13 times over $[0, 1]$.

These properties are utilized to find the Bézier points and to satisfy the geometric conditions of the circular arc. For more on these topics, see [5, 7].

In this paper, we allow the angle θ to be as large as possible in order to approximate the largest circular arc with the specified error. Thereafter, this angle θ has to be scaled by a factor that is also combined with a reduction in the uniform error, see the conclusions and open problems' section.

3 The Bézier curve

We use the curve $p(t)$ in Bézier form of degree 6 as follows

$$p(t) = \sum_{i=0}^{6} p_i B_i^6(t) =: \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad 0 \le t \le 1, \tag{3}$$

where $p_0, p_1, p_2, p_3, p_4, p_5$ and p_6 are the Bézier points, and $B_0^6(t) = (1-t)^6$, $B_1^6(t) = 6t(1-t)^5$, $B_2^6(t) = 15t^2(1-t)^4$, $B_3^6(t) = 20t^3(1-t)^3$, $B_4^6(t) = 15t^4(1-t)^2, B_5^6(t) = 6t^5(1-t)$ and $B_6^6(t) = t^6$ are the Bernstein polynomials of degree 6. Since our purpose is to represent the arc with a polynomial curve with the least possible error, it is not substantial for the errors to take place at the endpoints or elsewhere; it is significant to ensure that this error is as low as possible no matter where the error occurs.

We want to minimize the error over all of the interval $[0, 1]$. To explore the Bézier form approximation of a circular arc, a careful selection of locations of the 7 Bézier points should be well-done. These locations are substantial to earn the convenient curve that satisfies the approximation requirements. Because of the symmetry of the circle, the proper choice for the beginning control point p_0 should obey the following form: $p_0 = (\alpha_0 \cos(\theta),\ \beta_0 \sin(\theta))$, where values of α_0 and β_0 could but should not be the same. Similarly, for symmetry reasoning, the valid option for the end control point p_6 is $p_6 = (\alpha_0 \cos(\theta),\ -\beta_0 \sin(\theta))$. Set $p_1 = (a_1,\ b_1)$, then the point p_5 has to be selected to satisfy the form $p_5 = (a_1,\ -b_1)$. Set the point $p_2 = (a_2,\ b_2)$, then the point p_4 has to be selected to satisfy the form $p_4 = (a_2,\ -b_2)$. For symmetry purposes the point p_3 has to be selected to satisfy the form $p_3 = (a_3,\ 0)$. To simplify the notations, we set $a_0 = \alpha_0 \cos(\theta)$, $b_0 = \beta_0 \sin(\theta)$. So, the Bézier points, see Fig. 2, have the following form:

$$p_0 = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \ p_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \ p_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \ p_3 = \begin{pmatrix} a_3 \\ 0 \end{pmatrix},$$

$$p_4 = \begin{pmatrix} a_2 \\ -b_2 \end{pmatrix}, \ p_5 = \begin{pmatrix} a_1 \\ -b_1 \end{pmatrix}, \ p_6 = \begin{pmatrix} a_0 \\ -b_0 \end{pmatrix}. \tag{4}$$

We will find that there are more than one solution. The solution of best uniform approximation begins in the second quadrant and ends in the fourth quadrant counter clockwise. Therefore, in order to have the Bézier curve p begin in the second quadrant, go counter clockwise through fourth, third, first, second, and ends in the fourth quadrant as the circular arc c, the following stipulations should be satisfied:

$$a_0, a_1, b_1, b_2 < 0, \quad a_2, a_3, b_0 > 0. \tag{5}$$

Substitute the Bézier points in (5) in the Bézier curve $p(t)$ in (4) gives the following equation:

$$p(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$= \begin{pmatrix} a_0\left(B_0^6(t) + B_6^6(t)\right) + a_1\left(B_1^6(t) + B_5^6(t)\right) + a_2\left(B_2^6(t) + B_4^6(t)\right) + a_3 B_3^6(t) \\ b_0\left(B_0^6(t) - B_6^6(t)\right) + b_1\left(B_1^6(t) - B_5^6(t)\right) + b_2\left(B_2^6(t) - B_4^6(t)\right) \end{pmatrix}. \tag{6}$$

The Bézier curve is determined by the 7 parameters $a_0, a_1, a_2, a_3, b_0, b_1, b_2$. These parameters are used to get the best approximation with least deviation. We want to impose the requirements on the polynomial curve p; the polynomials $x(t)$ and $y(t)$ are substituted into $e(t)$. This leads to a polynomial of degree 12 that is solved using a computer algebra system; this is done in the next section.

4 The sextic Bézier curve of least deviation

In the following theorem, we give the values of $a_0, a_1, a_2, a_3, b_0, b_1, b_2$ that satisfy the requirements of the approximation problem.

Theorem 1: The Bézier curve in (6) determined by the Bézier points in (4) and the following values of the parameters:

$$a_0 = -0.6555549357237914, \ a_1 = -2.017393630688113, \ a_2 = 0.04396404726686853, \tag{7}$$

$$a_3 = 4.410826755740794, \ b_0 = 0.7554707191533404, \ b_1 = -0.44176804233980593, \tag{8}$$

$$b_2 = -3.595471196239275 \tag{9}$$

satisfies the following requirements: p minimizes the uniform norm of the error function $\max_{t\in[0,1]} |e(t)|$, and the error function $e(t)$ equioscillates 13 times in $[0,1]$, and the error functions satisfy for all $t \in [0,1]$:

$$-\frac{1}{2^{11}} \le e(t) \le \frac{1}{2^{11}},\ -\frac{1}{2^{11}(2-\epsilon)} \le E(t) \le \frac{1}{2^{11}(2+\epsilon)},\ \text{where } \epsilon = \max_{0\le t\le 1}|E(t)| \approx 2^{-12}.\ (10)$$

Proof: We substitute the polynomials $x(t)$ and $y(t)$ from equation (6) into the error function $e(t)$ in (2). Doing some algebraic manipulations gives the following equation:

$$\begin{aligned}
e(t) &= 4(a_0-6a_1+15a_2-10a_3)^2t^{12}-24(a_0-6a_1+15a_2-10a_3)^2t^{11}+12\left(8a_0^2+228a_1^2\right.\\
&\quad -1050a_1a_2+1200a_2^2+680a_1a_3-1550a_2a_3+500a_3^2-2a_0(43a_1-100a_2+65a_3)\\
&+ \left.3b_0^2-24b_0b_1+48b_1^2+30b_0b_2-120b_1b_2+75b_2^2\right)t^{10}\\
&- 20\left(13a_0^2+288a_1^2+1125a_2^2-90a_1(13a_2-8a_3)-1350a_2a_3+400a_3^2-2a_0(63a_1-\right.\\
&\quad \left.135a_2+85a_3)+9b_0^2-72b_0b_1+144b_1^2+90b_0b_2-360b_1b_2+225b_2^2\right)t^9\\
&+ 15\left(35a_0^2+576a_1^2-1992a_1a_2+1515a_2^2+a_0(-296a_1+562a_2-336a_3)+1136a_1a_3\right.\\
&- \left.1600a_2a_3+.400a_3^2+31b_0^2-232b_0b_1+432b_1^2+278b_0b_2-1032b_1b_2+615b_2^2\right)t^8\\
&- 12\left(67a_0^2+792a_1^2+a_0(-484a_1+770a_2-420a_3)-20a_1(111a_2-56a_3)+5\left(255a_2^2\right.\right.\\
&- \left.\left.220a_2a_3+40a_3^2+13b_0^2-88b_0b_1+144b_1^2+98b_0b_2-312b_1b_2+165b_2^2\right)\right)t^7\\
&+ 2\left(463a_0^2+3816a_1^2-8190a_1a_2+3375a_2^2+3360a_1a_3-2100a_2a_3+200a_3^2-42a_0(67a_1\right.\\
&- \left.85a_2+40a_3)+461b_0^2-2730b_0b_1+3744b_1^2+2730b_0b_2-6930b_1b_2+2925b_2^2\right)t^6\\
&- 12\left(66a_0^2+360a_1^2-555a_1a_2+150a_2^2+a_0(-331a_1+40(8a_2-3a_3))+160a_1a_3\right.\\
&- \left.50a_2a_3+66b_0^2-329b_0b_1+360b_1^2+280b_0b_2-525b_1b_2+150b_2^2\right)t^5\\
&+ 15\left(33a_0^2+108a_1^2-108a_1a_2+15a_2^2+16a_1a_3-4a_0(33a_1-23a_2+6a_3)+33b_0^2\right.\\
&- \left.132b_0b_1+108b_1^2+88b_0b_2-108b_1b_2+15b_2^2\right)t^4\\
&- 20\left(11a_0^2+18a_1^2-9a_1a_2+a_0(-33a_1+15a_2-2a_3)+11b_0^2-33b_0b_1+18b_1^2\right.\\
&+ \left.15b_0b_2-9b_1b_2\right)t^3\\
&+ 6\left(11a_0^2-22a_0a_1+6a_1^2+5a_0a_2+11b_0^2-22b_0b_1+6b_1^2+5b_0b_2\right)t^2\\
&- 12\left(a_0^2-a_0a_1+b_0(b_0-b_1)\right)t-1+a_0^2+b_0^2.
\end{aligned}$$

The approximation requirements are fulfilled if the error function is equal to the polynomial of least deviation of degree 12. So, the last equation which exemplifies the error function has to be equalized with the Chebyshev polynomial of first kind of degree 12, $\tilde{T}_{12}(2t-1)/2048$. The monic Chebyshev polynomial of degree 12,

$T_{12}(u) = \cos(12 \arccos(u)),\ u \in [-1, 1]$ is the unique monic polynomial of degree 12 that has the least uniform error. Equating the coefficients of the same powers on both sides of the equality, then the solution that satisfies the requirements of the approximation problem in (5) is found. These solutions are given in equations (7) - (9) and thus, it is proved that p satisfies the requirements of the approximation problem. We know that

$$E(t) = \frac{e(t)}{2 + E(t)}.$$

this relation leads to the following inequalities:

$$-\frac{1}{2^{11}(2-\epsilon)} \leq E(t) \leq \frac{1}{2^{11}(2+\epsilon)}, \quad \text{where} \ \ \epsilon = \max_{0 \leq t \leq 1} |E(t)| \approx 2^{-12}, \quad t \in [0, 1].$$

This proves the results stated in Theorem 1. □

The the approximating Bézier curve of degree six and the circular arc are visualized in Fig. 2. The resulting error between the curve and the approximation is not identified by the human eyes which is clear from figure of the corresponding error plotted in Fig. 3.

The characteristics of the approximating Bézier curve are specified in the following section.

5 Properties of the sextic Bézier curve

The most important characteristics of the error functions are the roots and the extrema. These properties characterize the approximating sextic Bézier curve. The first characteristic concerns the roots of the error functions $e(t)$ and $E(t)$ that are specified in the following proposition.

Proposition I: The roots of the error functions $e(t)$ and $E(t)$ are:

$$t_1 = \frac{1}{2}(1+\cos(\frac{\pi}{24})) = 0.995722, t_2 = \frac{1}{2}(1+\cos(\frac{3\pi}{24})) = 0.96194, t_3 = \frac{1}{2}(1+\cos(\frac{5\pi}{24})) = 0.896677,$$

$$t_4 = \frac{1}{2}(1+\sin(\frac{5\pi}{24})) = 0.804381, t_5 = \frac{1}{2}(1+\sin(\frac{3\pi}{24})) = 0.691342, t_6 = \frac{1}{2}(1+\sin(\frac{\pi}{24})) = 0.565263,$$

$$t_7 = \frac{1}{2}(1-\sin(\frac{\pi}{24})) = 0.434737, t_8 = \frac{1}{2}(1-\sin(\frac{3\pi}{24})) = 0.308658, t_9 = \frac{1}{2}(1-\sin(\frac{5\pi}{24})) = 0.195619$$

$$t_{10} = \frac{1}{2}(1-\cos(\frac{5\pi}{24})) = 0.103323, t_{11} = \frac{1}{2}(1-\cos(\frac{3\pi}{24})) = 0.0380602, t_{12} = \frac{1}{2}(1-\cos(\frac{\pi}{24})) = 0.00427757$$

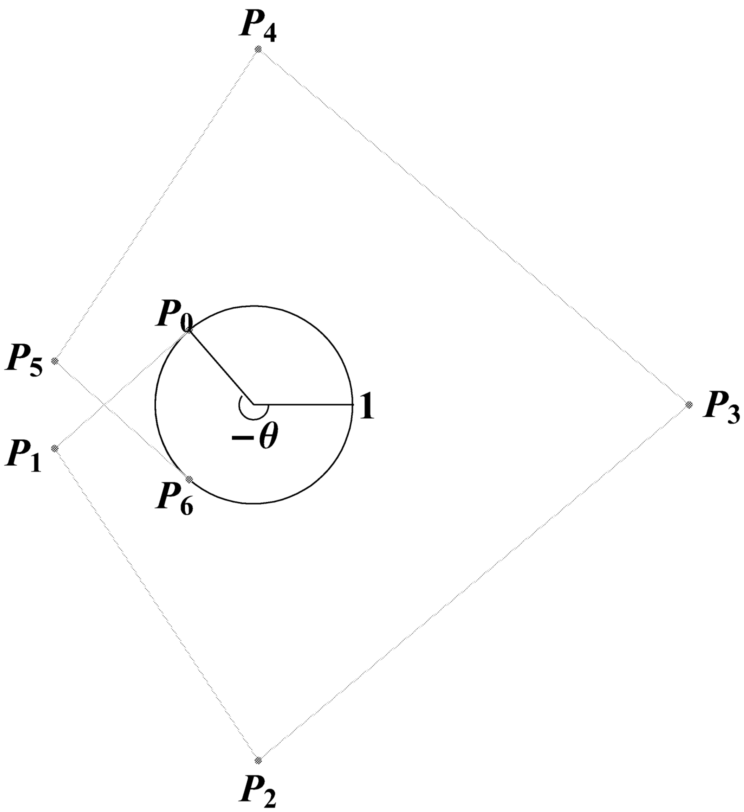

Figure 2: Circular arc and it's sextic Bézier curve in Theorem 1.

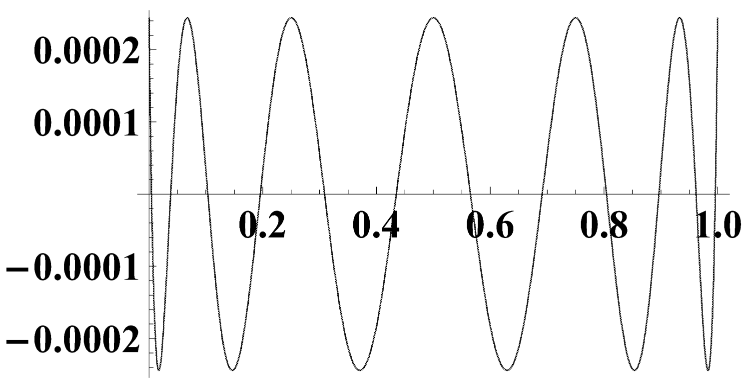

Figure 3: Euclidean Error of the sextic Bézier curve in Theorem 1.

They also satisfy

$$t_i + t_j = 1, \quad \text{for} \quad i + j = 13.$$

Proof: Immediate substitution of the values of t_i in $e(t)$ gives $e(t_i) = 0$, $i = 1, 2, \ldots, 12$. Since $e(t)$ is a polynomial of degree 12 and has 12 roots, therefore, these ones are all the roots. The error function $E(t)$ has the same roots as $e(t)$ because $E(t) = 0$ iff $\sqrt{x^2(t) + y^2(t)} = 1$ iff $x^2(t) + y^2(t) = 1$ iff $e(t) = 0$. □

The approximating sextic Bézier curve p in Theorem 1 and the circular arc c intersect at the points $p(t_i) = c(t_i)$, $i = 1, 2, \ldots, 12$.

Regarding the extreme values, we have the following proposition.

Proposition II: The extreme values of $e(t)$ and $E(t)$ occur at the parameters:

$$\tilde{t}_0 = 1, \quad \tilde{t}_1 = \frac{1}{2}(1 + \cos(\frac{\pi}{12})) = 0.982963, \quad \tilde{t}_2 = \frac{1}{2}(1 + \cos(\frac{\pi}{6})) = 0.933013,$$

$$\tilde{t}_3 = \frac{1}{2}(1 + \cos(\frac{\pi}{4})) = 0.853553, \quad \tilde{t}_4 = \frac{1}{2}(1 + \cos(\frac{\pi}{3})) = 0.75,$$

$$\tilde{t}_5 = \frac{1}{2}(1 + \cos(\frac{5\pi}{12})) = 0.62941, \quad \tilde{t}_6 = \frac{1}{2}, \quad \tilde{t}_7 = \frac{1}{2}(1 - \cos(\frac{5\pi}{12})) = 0.37059,$$

$$\tilde{t}_8 = \frac{1}{2}(1 - \cos(\frac{\pi}{3})) = 0.25, \ \tilde{t}_9 = \frac{1}{2}(1 - \cos(\frac{\pi}{4})) = 0.146447,$$

$$\tilde{t}_{10} = \frac{1}{2}(1 - \cos(\frac{\pi}{6})) = 0.0669873, \ \tilde{t}_{11} = \frac{1}{2}(1 - \cos(\frac{\pi}{12})) = 0.0170371, \quad \tilde{t}_{12} = 0.$$

These parameters satisfy the equality:

$$\tilde{t}_i + \tilde{t}_j = 1, \quad \text{for} \quad i + j = 12.$$

Proof: The derivative of $e(t)$ is a polynomial of degree 11. Substituting the 11 parameters $\tilde{t}_1, \ldots, \tilde{t}_{11}$ into this derivative gives $e'(\tilde{t}_i) = 0$, $\forall i = 1, \ldots, 11$. The polynomial $e'(t)$ has degree 11 and consequently these are all internal critical points. Inspecting the end points adds $\tilde{t}_0 = 1$, $\tilde{t}_{12} = 0$ to the critical points. For all $t \in [0, 1]$, we have $1 - \frac{1}{2048} \le x^2(t) + y^2(t) \le 1 + \frac{1}{2048}$, thence $\sqrt{x^2(t) + y^2(t)} \ne 0$, $\forall t \in [0, 1]$. Differentiate $E(t)$ and counter equate to 0 to acquire $\frac{e'(t)}{\sqrt{x^2(t)+y^2(t)}} = 0$ iff $e'(t) = 0$. Therefore, $e(t)$ and $E(t)$ reach the extrema at the same values. This finishes the proof of the proposition. □

The disagreement in the values of $E(\tilde{t}_i)$ for odd and even i's occurs because $e(t)$ equioscillates between $\pm\frac{1}{2048}$ and $\frac{1}{2^{11}(2-\epsilon)} \le E(t) \le \frac{1}{2^{11}(2+\epsilon)}$, where $\epsilon = max_{0 \le t \le 1}|E(t)|$.

Proposition III: The values of the error functions $e(t)$ and $E(t)$ at $\tilde{t}_i$'s are specified by:

$$e(\tilde{t}_{2i}) = \frac{1}{2048}, i = 0, \ldots, 6, \qquad e(\tilde{t}_{2i+1}) = \frac{-1}{2048}, i = 0, \ldots, 5.$$
$$E(\tilde{t}_{2i}) = \frac{1}{4096}, i = 0, \ldots, 6, \qquad E(\tilde{t}_{2i+1}) = \frac{-1}{4096}, i = 0, \ldots, 5.$$

Therefore,

$$\frac{-1}{2048} \leq e(t) \leq \frac{1}{2048}, \quad \frac{-1}{4096} \leq E(t) \leq \frac{1}{4096}, \ t \in [0, 1].$$

Proof: Substituting the parameters in the error functions confer to the parities. The specifics of the proof of the proposition are left to the reader. □

The following proposition is a conclusion of Theorem 1 concerning the error at any $t \in [0, 1]$.

Proposition IV: The errors of approximating the circular arc using the sextic Bézier curve in Theorem 1 at any $t \in [0, 1]$ are given by:

$$e(t) = \frac{1}{2048} - \frac{9}{64}t + \frac{429}{64}t^2 - \frac{1001}{8}t^3 + \frac{19305}{16}t^4 - 6864t^5 + 24752t^6 - 58752t^7 + 93024t^8$$
$$-97280t^9 + 64512t^{10} - 24576t^{11} + 4096t^{12}, \quad \forall t \in [0, 1].$$

Proof: This is a forthright conclusion of Theorem 1. The specifics of the proof of the proposition are left to the reader. □

Employ the relation between $E(t)$ and $e(t)$ to obtain:

$$E(t) \tilde{=} \frac{1}{4096} - \frac{9}{128}t + \frac{429}{128}t^2 - \frac{1001}{16}t^3 + \frac{19305}{32}t^4 - 3432t^5 + 12376t^6 - 29376t^7 + 46512t^8$$
$$-48640t^9 + 32256t^{10} - 12288t^{11} + 2048t^{12}, \quad \forall t \in [0, 1].$$

6 Conclusions and Open Problems

The sextic approximation of the circle is presented in this paper. The classical approximation guarantees that the error function equioscillates 8 times; but there is no method to find this approximation. The approximation in this paper equioscillates 13 times with the Chebyshev polynomial as the error function that can not be made better. The Bézier curve intersects the circular arc 12 times with maximum error 2^{-11}.

Our method has the following advantages:

1. It beats any other approximation to the circular arc with polynomials of degree 6
2. The method is convenient also for curves and not only for functions.
3. The posture of Bézier points of the resulting Bézier curve is masterful. This helps designers using the Bézier curves to better understand choosing the Bézier points.

Possible further investigations can be taken to solve the following issues:

1. Is it possible to find the polynomial of degree n of best uniform approximation, using a similar method to other kinds of curves, that equioscillates $2n+1$ times rather than classical $n+2$ times.
2. Study sextic approximation with G^k- continuity using equioscillating error functions and constrained Chebyshev polynomials.
3. Detect a relation to write the Bézier points using the angle θ. This authorizes us to prescribe the best uniform approximation for arbitrary circular arc.
4. Apply the approximation in this paper to implement degree reduction of Bézier curves to gain the best approximation with the least possible error.

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